HOMEWORK SET 0: REMEMBERING MATH

Due: Monday, September 2, 2019

Perform the following operations:

$$\frac{d}{dt}e^{-kt} = -ke^{-kt} \qquad \frac{d}{dt}\frac{1}{kt} = \frac{-1}{kt^2} \qquad \frac{d}{dt}\ln(kt) = \frac{k}{kt} = \frac{1}{t}$$

$$\int e^{-kt}dt = \frac{-1}{k}e^{-kt} \qquad \int\frac{dt}{kt} = \frac{1}{k}\ln(kt) \qquad \int\ln(t)dt = t\ln(t) - t$$

#2b $$\int\frac{dt}{1+kt} = \frac{1}{k}\ln(1+kt) \qquad \left\{\frac{d}{dt}\left[\ln(1+kt)\right] = \frac{k}{1+kt}\right.$$

#2a $$\int\frac{tdt}{1+kt} = \frac{1}{k}\int\frac{kt+1-1}{1+kt}dt = \frac{1}{k}\left[\int dt + \int\frac{dt}{1+kt}\right] = \frac{t}{k} - \frac{1}{k^2}\ln(1+kt)$$

$$\int\frac{tdt}{1+kt^2} = \frac{1}{2k}\int\frac{2kt\,dt}{1+kt^2} = \frac{1}{2k}\ln\left(1+kt^2\right)$$

The hyperbolic functions are defined as

$$\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right) \qquad \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right)$$

Show that

$\Rightarrow z = \sinh(w) = \frac{1}{2}(e^{w} - e^{-w})$

$$\sinh^{-1}(z) = w \quad \Rightarrow \quad \sinh^{-1}(z) = \ln\left(z + \sqrt{z^2+1}\right)$$

(hint: start with $2z = e^{w} - e^{-w}$ and solve for w … you'll have to solve a quadratic in e^{2w} (& e^{w} & 1) and note that since the radical is always ±, technically, ±√ = +√)

$$(2z = e^{w} - e^{-w})e^{w} \Rightarrow 2ze^{w} = e^{2w} - 1 \Rightarrow \underbrace{e^{2w} - 2ze^{w} - 1 = 0}_{\text{APPLY QUADRATIC TO } e^{w}!}$$

$$e^{w} = \frac{2z \pm \sqrt{4z^2 - 4(1)(-1)}}{2} = \frac{\cancel{2}z \pm \cancel{2}\sqrt{z+1}}{\cancel{2}}$$

$$\underbrace{e^{w} = z \pm \sqrt{z^2-1}}_{\text{SQUARE ROOTS ARE ALWAYS } \pm \text{ SO JUST WRITE "+"}} \Rightarrow \boxed{w = \sinh^{-1}(z) = \ln\left(z + \sqrt{z^2+1}\right)}$$

QED!

Then show that

$$\frac{d}{dz}\ln\left(z + \sqrt{z^2+1}\right) = \frac{1}{\sqrt{z^2+1}}$$

$$= \frac{1 + \frac{1}{\cancel{2}}(z^2+1)^{-1/2}(\cancel{2}z)}{z + \sqrt{z^2+1}} =$$

$$= \left(\frac{1}{z + \sqrt{z^2+1}}\right)\left(1 + \frac{z}{\sqrt{z^2+1}}\right) = \left(\frac{1}{\cancel{z + \sqrt{z^2+1}}}\right)\left(\frac{\cancel{\sqrt{z^2+1} + z}}{\sqrt{z^2+1}}\right)$$

$$\Rightarrow \boxed{\frac{d}{dz}\ln\left(z + \sqrt{z^2+1}\right) = \frac{1}{\sqrt{z^2+1}}}$$

QED!

Then show that (use the table of derivatives in the Pocket Book of Integrals and Mathematical Formulas)

$$\frac{d}{dz}\cosh^{-1}\left(e^{kz}\right) = \frac{ke^{kz}}{\sqrt{e^{2kz}-1}} = \frac{k}{\sqrt{1-e^{-2kz}}}$$

QED!

$$\frac{ke^{kz}}{\sqrt{e^{2kz}-1}}\left(\frac{e^{-kz}}{e^{-kz}}\right) = \frac{k}{\sqrt{(e^{-2kz})(e^{2kz}-1)}} = \frac{k}{\sqrt{1-e^{-2kz}}}$$